Degrees of freedom in rank deficient Channels

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Abstract

In this report we will briefly discuss about degrees of freedom in the Interference channels with rank deficiency. The rank deficiency can be in the individual point to point channels of Interference model, like MIMO Channels, or in the entire system, like Interfering SISO Channels. For the case of rank deficiency in the point to point MIMO Interfering Channels, the achievability of optimal dof has been established while the same has not been done so far now for the SISO case. We will see some observed difficulties in the existing work which tried to achieve optimal dof in the SISO case.

Introduction

Degrees of freedom (dof) is the pre-log factor which appears in the Channel capacity expression. In other words dof determines the number of parallel (independent) channels in which we can decompose the complex wireless channel. Therefore, for a point to point MIMO channel we can say that the dof is equal to rank of the channel matrix. When inter-dependencies arise in the wireless channel like absence of rich scattering or correlation between the antennas, the rank deficiency occur in the Channel. Interestingly, if we consider a network topology in which we have given end-to-end transmitters and receivers but not the internal structure which includes relays or routers, switches and somewhat unknown network connections between them. This network structure can be looked in the wireless domain as a SISO interfering channel but with individual channels not being entirely independent or in other words, some deficiency in the rank of the overall channel. It has

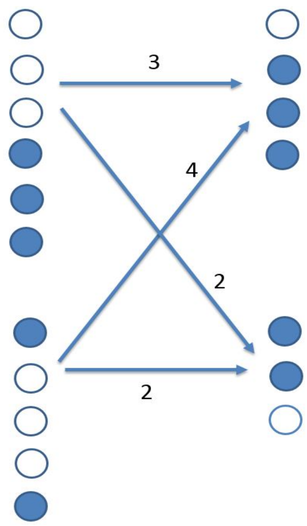

Figure 1: 2 user Rank deficient MIMO IC

been already established that the dof of Interfering Channel (IC) with full rank has a maximum value of $K/2$. While the achievability of the same optimal dof for the rank deficient case is still an open problem. We will appreciate this problem by starting from MIMO rank deficient IC and then moving on to the SISO IC case.

Rank deficient MIMO IC

The optimal dof of rank deficient MIMO IC with 2 and 3 user case with asymmetric setting has been established in [1] and it is generalized for $K-$ user case using asymptotic interference alignment in [2]. We will touch upon some ideas involved in achieving optimal dof in the case of 2 and 3 user case.

Consider a 2 user example as shown in the Fig. 1 where we have taken 6 antennas at transmitter T1 and 5 antennas at another transmitter, T2. We have 2 receivers R1, R2 with 4 and 3 antennas respectively. The individual channels are rank deficient and the corresponding ranks are as shown in the figure. The transmitter T1 is going to make first 3 inputs 0 and transmit rest 3 because rank of the direct channel is 3. The receiver R1 is going to discard the first output and use only last 3 antennas. The transmitter T2 is going to make $4 - 1 = 3$ input entries 0 in order to cause no interference at the receiver R1. This can be done by setting the inputs from 2 to 4 as 0. The first output of the receiver R1 is going to be discarded anyways, hence T2 can transmit in the first dimension. Thus, R1 will now have 3 interference free dimensions and its dof would be 3. For the receiver R2, since the transmitter T1 is not causing any interference in the first 2 dimensions as the rank of this interference channel itself is 2, we will have 2 interference free dimensions at the receiver R2 and its dof would be 2. Therefore, in this system we will have a total dof of $3 + 2 = 5$.

In [1], a general expression has been provided for any transmitter-receiver antenna combination and the ranks of the channels. The achievable optimal bound is written as

$$\text{dof} \leq \min\{d_{11} + d_{22}, M_1 + N_2 - d_{21}, M_2 + N_1 - d_{12}\}$$

where M_i, N_i is the number of antennas at the ith transmitter, receiver and d_{ij} is rank of the channel from jth transmitter to the ith receiver. In our example we have

$$\begin{aligned} \text{dof} &\leq \min\{2+3, 6+3-2, 5+4-4\} \\ &= 5 \end{aligned}$$

which is what we have seen in the discussed achievable scheme. Next, we will discuss how achievability is established for the optimal dof in 3-user case.

Let us consider a case in which every node in the system has equal number of antennas, M, and all the direct channels have rank of d_0 and the cross channels of the form $\mathbf{H}_{k(k+1)}, \mathbf{H}_{k(k-1)}$ have rank d_1, d_2 respectively. The achievability of the optimal dof is based on the design of the transmit vectors which depends upon the available dimensions decided by the variation of rank of the channels in comparison of M.

At each receiver we will have a desired signal and two interference terms. The transmit vectors are therefore designed to either cancel the interference or to align the interference so that the effective interference dimensions can be reduced. The transmit vectors at the kth transmitter can be classified into four categories namely

1. $\mathbf{V}_k^{Za}$ is the trasmit vector which lies in the null space of the $\mathbf{H}_{(k-1)k}$ and hence it will cancel the interference received at the $(k-1)$th receiver from kth transmitter.

2. $\mathbf{V}_k^{Zb}$ lies in null space of the $\mathbf{H}_{(k+1)k}$ and hence it will eliminate the interference received at $(k+1)$th receiver from kth transmitter

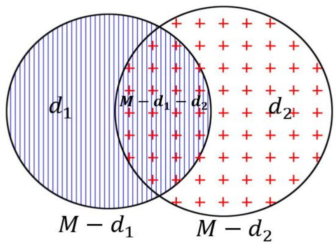

Figure 2: Overlap of Null spaces

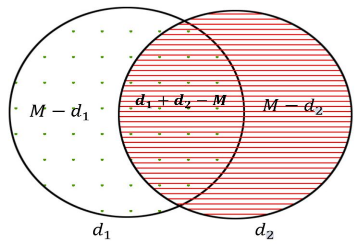

Figure 3: Overlap of Interference spaces

3. $\mathbf{V}_k^{Zc}$ lies in the overlap of null spaces of $\mathbf{H}_{(k+1)k}$ and $\mathbf{H}_{(k-1)k}$, hence unlike above two vectors which eliminate interference at one receiver but cause at another, this transmit vector will not cause any interference at any receiver. This beamformer may or may not exist depending on rank of the interfering channels.

4. $\mathbf{V}_k^{A}$ is responsible for aligning the interference spaces at the receiver so that the effective dimensions occupied can be reduced. The existence of alignment vectors is also governed by rank of the interfering channels. We can say that if $d_1 + d_2 > M$ then we will have overlap in the interfering channel spaces and hence it is possible to design these vectors.

It can be easily seen that dimension of the desired signal is going to be $\dim(\mathbf{V}_k^{Za})+\dim)(\mathbf{V}_k^{Zb})+\dim)(\mathbf{V}_k^{Zc})+\dim)(\mathbf{V}_k^{A})$ while dimension of interference is same as desired signal expression except for the $\dim)(\mathbf{V}_k^{Zc})$ term as $\mathbf{V}_k^{Zc}$ by definition does not cause interference at either of the receiver. While deciding the dimension for these transmit vectors it has always been taken care that the sum dimensions of the interference and desired signal is always equal to M so that they can be separated at the receiver. A variety of cases are possible depending upon the values of $d_1 + d_2$ as compared to M.

The design of the zero-forcing vectors is as follows: For a d_1 dimensional interference channel, there will be $M - d_1$ null space dimensions and hence $\dim(\mathbf{V}_k^{Za}) \leq M - d_1$, similarly $\dim(\mathbf{V}_k^{Zb}) \leq M - d_2$. When the value of $d_1 + d_2 \leq M$ then the overlap dimension of two $(M - d_1)$ and $(M - d_2)$ dimensional nullspaces will be $(M - d_1 - d_2)$ which is the maximum possible dimension of $\mathbf{V}_k^{Zc}$. For the case when $d_1 + d_2 > M$ there will be no overlap of null spaces hence $\dim(\mathbf{V}_k^{Zc}) = 0$ but there will be an overlap in the interference spaces of dimensions d_1 and d_2 as $(d_1 + d_2 - M)$ which is what the maximum possible value of $\dim(\mathbf{V}_k^{A})$.

The alignment vectors are designed to exploit the overlap of the interference spaces to reduce the effective interference dimensions. A certain amount of interference from one transmitter can be shadowed by the interference from other transmitter if they are aligned to span the same subspace. The above statement can be written mathematically as

$$\begin{aligned}
\text{span}(\mathbf{H}_{21}\mathbf{V}_1^A) &\subseteq \text{span}(\mathbf{H}_{32}\mathbf{V}_2)\\
\text{span}(\mathbf{H}_{32}\mathbf{V}_2^A) &\subseteq \text{span}(\mathbf{H}_{31}\mathbf{V}_1)\\
\text{span}(\mathbf{H}_{13}\mathbf{V}_3^A) &\subseteq \text{span}(\mathbf{H}_{12}\mathbf{V}_2)
\end{aligned}$$

where we can see that the interference dimensions beamformed by $\mathbf{V}_k^A$ always span the same subspace as spanned by the interference from other transmitter and thus will cause no more harm. This scheme achieves the optimal dof for 3 user case and the outerbound can be written as

$$\text{dof} \leq \min\left\{d_0, M - \frac{\min(M, d_1 + d_2)}{2}\right\}$$

Rank deficient SISO IC

While we have seen the case of rank deficiency in the individual MIMO channels, however, rank deficiency can also arise in the combined SISO IC. Such deficiency will create dependencies between direct and cross channels of the users in the system. Consider a K user SISO IC model in which rank of the IC channel is d where $d < K$. While this problem is not solved till now entirely, but [3] has moved in this direction and showed some difficulties in achieving the outer bound of $K/2$. We will see some of its results but before that lets introduce some mathematical preliminaries.

Variety:

We have a set of multivariate polynomials and rational functions in the variables $t_1, t_2, ...t_n$, which we denote as $\mathbb{C}[t_1, t_2, ..., t_n]$ and $\mathbb{C}(t_1, t_2, ..., t_n)$ respectively. For any polynomials $f_1, f_2, ..., f_m \in \mathbb{C}[t_1, t_2, ..., t_n]$, the affine variety generated by $f_1, f_2, ..., f_m$ is denoted by set of points at which the polynomials vanish i.e.

$$V(\mathbf{f}) = \{\mathbf{t} \in \mathbb{C}^n : \mathbf{f}(\mathbf{t}) = \mathbf{0}\}$$

Ideal

A subset I of $\mathbb{C}[t_1, t_2, ..., t_n]$ is called an ideal if it has the following properties:

1. $0 \in I$.
2. If $f_1, f_2 \in I$, then $f_1 + f_2 \in I$.
3. If $f_1 \in I$ and $f_2 \in \mathbb{C}[t_1, t_2, ..., t_n]$, then $f_1 f_2 \in I$.

Likewise, for any set $\mathcal{A} \subseteq \mathbb{C}^n$, the ideal generated by $\mathcal{A}$ is defined as

$$I(\mathcal{A}) = \{f \in \mathbb{C}[t_1, t_2, ..., t_n] : f(\mathbf{t}) = 0, \forall \mathbf{t} \in \mathcal{A}\}$$

For any ideal I, the affine variety generated by I is defined as

$$V(I) = \{\mathbf{t} \in \mathbb{C}^n : f(\mathbf{t}) = 0 \forall f \in I\}$$

Having this knowledge we will see that how dof bounds can be proved for any channel rank $d \neq \lceil K/2 \rceil$. From here we will assume that dof outerbound of $\lceil K/2 \rceil$ is achievable if $d = \lceil K/2 \rceil$. First consider the case of $d < \lceil K/2 \rceil$. This implies that we can select $2d$ users out of K and from our assumption we can achieve dof of $2d/2 = d$ in this system. A symmetric dof for each user can be achieved if we cycle the subsets of users. We will have a total possibilities of $\binom{K}{2d}$ out of which a particular user is going to be active in $\binom{K-1}{2d-1}$ and achieves $1/2$ dof. Therefore, overall dof achieved by a particular user through all cycles in going to be

$$dof = \frac{1}{2} \times \frac{\binom{K-1}{2d-1}}{\binom{K}{2d}} = \frac{d}{K}$$

Next consider the case when $d > \lceil K/2 \rceil$. It has been shown in [3] that if the direct channels cannot be expressed as rational polynomial functions of cross channel terms, then by using asymptotic interference alignment the dof outerbound of $K/2$ can be achieved, arbitrarily close. Moving forward, we know that for a rank d channel matrix we will have the value of determinant of all the $(d+1) \times (d+1)$ matrices 0. Using concepts of variety and ideals we can say that these determinant polynomials generate an affine variety, V_d. Without causing any harm to our arguments, it can be assumed that $d = \lceil K/2 \rceil + 1$. Therefore, we have a variety $V_{\lceil K/2 \rceil + 1}$ and an ideal generated by this variety, $I_{\lceil K/2 \rceil + 1} = I(V_{\lceil K/2 \rceil + 1})$. It can be said that

$$V_{\lceil K/2 \rceil + 1} \subseteq V_{\lceil K/2 \rceil}$$

which in turn implies that

$$I(V_{\lceil K/2 \rceil}) \supseteq I(V_{\lceil K/2 \rceil + 1})$$

It can be argued that since we have taken $d = \lceil K/2 \rceil + 1$, we can express direct channels as rational function of polynomials of cross channels, or in other words we have a polynomial f_1, $A(\mathbf{S})H_{kk} - B(\mathbf{S})$ which always evaluates to 0. Here, $\mathbf{S}$ denotes set of cross-channel terms and $A(.), B(.)$ are some polynomial functions. Since $f_1 = 0$ under affine variety $V_{\lceil K/2 \rceil + 1}$, therefore $f_1 \in I_{\lceil K/2 \rceil + 1}$ and hence $f_1 \in I_{\lceil K/2 \rceil}$. But this contradicts the assumption that we can achieve a dof of $\lceil K/2 \rceil$ when $d = \lceil K/2 \rceil$. Thus we can say that for channels with rank $d = \lceil K/2 \rceil + 1$, the achievable dof cannot exceed $\lceil K/2 \rceil$. Similar arguments can be applied for all $d > \lceil K/2 \rceil$ and we can finally write the dof expression for any rank d as

$$dof_K(d) \leq \min(d, \lceil K/2 \rceil)$$

Now, that we will come to the main problem of achieving dof of $\lceil K/2 \rceil$ when rank, $d = \lceil K/2 \rceil$. We will first try to see that if this is even achievable for some channel combinations. Let us start this discussion by stating that a generic $K \times K$ Gaussian rank deficient channel, $\mathbf{H}$ can be mathematically expressed as

$$\mathbf{H} = \mathbf{F} \times \mathbf{G}$$

where $\mathbf{F}$ and $\mathbf{G}$ are $K \times d$ and $d \times K$ matrices with entries from circularly symmetric Gaussian distribution. To start with, we will consider time varying channels and take two time slots. If each user can achieve 1 dof in these 2 time-slots then we will have a dof of 1/2 per user. In the first time-slot we have channel matrix as $\mathbf{H}_1 = \mathbf{F}_1\mathbf{G}_1$ and the received signal at the kth receiver can be written as

$$y_1^{[k]} = \mathbf{h}_1^{[k]\,H}\mathbf{X} + n_1^{[k]}$$

where $\mathbf{h}_1^{[k]\,H}$ is the kth row vector of $\mathbf{H}_1$ matrix and $\mathbf{X}$ is $K \times 1$ transmit vector, $\mathbf{X} = [x_1^H, x_2^H, ..., x_K^H]^H$ with x_k as desired symbol for the kth receiver. Similarly, for the second time slot we can write

$$y_2^{[k]} = \mathbf{h}_2^{[k]\,H}\mathbf{X} + n_2^{[k]}$$

In order to achieve dof of 1 in these 2 time-slots, the receiver should be able to eliminate the interference free from linear combination of these 2 receptions. Therefore, we can write for the first user as

$$
\begin{aligned}
y_1^{[1]} + \lambda_1 y_1^{[2]} &= \mathbf{h}_1^{[1]\,H}\mathbf{X} + \lambda_1 \mathbf{h}_2^{[1]\,H}\mathbf{X} + n_1^{[1]} + \lambda_1 n_2^{[1]} \\
&= (\mathbf{h}_1^{[1]\,H} + \lambda_1 \mathbf{h}_2^{[1]\,H})\mathbf{X} + \tilde{n}^{[1]} \\
&\overset{(a)}{=} \alpha_1 \mathbf{e}_1^H \mathbf{X} + \tilde{n}^{[1]}
\end{aligned}
$$

where $\mathbf{e}_j$ is the standard unit vector with all zeros except at the jth place and we have written (a) because of the requirement of the user-1 to eliminate all interference from linear combination of two time-slots receptions and reduce it to the desired signal only, in this case x_1. The constant α_1 is effective channel gain after linear processing. We can write the similar equations for channel requirements for all the users as

$$
\begin{aligned}
\mathbf{h}_1^{[1]\,H} + \lambda_1 \mathbf{h}_2^{[1]\,H} &= \alpha_1 \mathbf{e}_1^H \\
\mathbf{h}_1^{[2]\,H} + \lambda_2 \mathbf{h}_2^{[2]\,H} &= \alpha_2 \mathbf{e}_2^H \\
&\vdots \\
\mathbf{h}_1^{[K]\,H} + \lambda_K \mathbf{h}_2^{[K]\,H} &= \alpha_K \mathbf{e}_K^H
\end{aligned}
$$

which can be represented in matrix form as

$$
\mathbf{H}_1 + \mathbf{\Lambda}\mathbf{H}_2 = \mathbf{A}
$$

where $\mathbf{\Lambda}$ and $\mathbf{A}$ are the diagonal matrices with entries λ_i and α_i, respectively. Using the decomposition of $\mathbf{H}_1$ and $\mathbf{H}_2$ we can further write

$$
\begin{bmatrix} \mathbf{F}_1 & \mathbf{\Lambda}\mathbf{F}_2 \end{bmatrix} \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix} = \mathbf{A}
$$

which implies

$$
\begin{bmatrix} \mathbf{F}_1 & \mathbf{\Lambda}\mathbf{F}_2 \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix}^{-1}
$$

Therefore, for a given random $\mathbf{\Lambda}$ and $\mathbf{A}$ we can first construct a full rank matrix $\mathbf{G}$, take its inverse to get right hand side of the above equation and correspondingly select $\mathbf{F}_1$ and $\mathbf{F}_2$. Thus we have a family of rank deficient channels through which every can achieve a dof of 1/2.

Conclusion

We have seen examples of rank deficiency in 2 and $3-$user MIMO IC and schemes which achieve the dof outerbounds. For the rank deficient SISO case we have seen that the dof outerbounds can be achieved if we assume that dof for rank, $d = \lceil K/2 \rceil$ is achievable. For this case of interest, we have seen that a family of channels exist which can achieve 1/2 dof per user with linear processing. A future work could be to explore schemes that can achieve the dof outerbound for generic rank $\lceil K/2 \rceil$ channel.

References

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